



## Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact [support@jstor.org](mailto:support@jstor.org).

# ON SOME PROPERTIES OF POLYNOMIALS.

BY E. L. DE FOREST.

(Continued from page 46.)

WE pass now to the consideration of that other property of polynomials of one variable, which was demonstrated in the *ANALYST* for Jan., 1880, pp. 5 to 7. It can be treated in a more simple and general way as follows.

Let the coefficients in the two polynomials, (17) of the present paper, be regarded as magnitudes placed at equal distances  $\Delta x$  along a straight line which is taken as the axis of  $X$ . If these two systems of coefficients were parallel forces acting as was supposed, they would have each a centre of parallel forces, located in the axis of  $X$ . As before, let the distances of these centres from the places of  $a_0$  and  $c_0$  be  $h_1$  and  $h_2$ , and let the sums of the coefficients in the two systems be  $S_1$  and  $S_2$ . Also let  $r_1$  and  $r_2$  be distances such that  $S_1 r_1^2$  in the first system is equal to the sum of the products formed by multiplying each coefficient of that system into the square of its distance from the centre of parallel forces, while  $S_2 r_2^2$  has a similar significance in the second system. These distances,  $r_1$  and  $r_2$  would be the radii of gyration, if the coefficients in each system were the masses of material points attached to the imponderable axis of  $X$ , and rotating in the plane of  $XY$  about the centre of parallel forces. They cannot however be completely represented as such because masses are always positive, while the coefficients in (17) may be either positive or negative. Still, for convenience of nomenclature, we will call  $r_1$  and  $r_2$  radii of gyration.

Now according to the definition,

$$\left. \begin{aligned} S_1 r_1^2 &= a_0 h_1^2 + a_1 (h_1 - \Delta x)^2 + a_2 (h_1 - 2\Delta x)^2 + \dots + a_m (h_1 - m\Delta x)^2, \\ S_2 r_2^2 &= c_0 h_2^2 + c_1 (h_2 - \Delta x)^2 + c_2 (h_2 - 2\Delta x)^2 + \dots + c_n (h_2 - n\Delta x)^2. \end{aligned} \right\} \quad (29)$$

Expanding the binomials and arranging the terms,

$$\left. \begin{aligned} S_1 r_1^2 &= h_1^2 (a_0 + a_1 + \dots + a_m) - 2h_1 \Delta x (a_1 + 2a_2 + \dots + ma_m) \\ &\quad + (\Delta x)^2 (1^2 a_1 + 2^2 a_2 + \dots + m^2 a_m), \\ S_2 r_2^2 &= h_2^2 (c_0 + c_1 + \dots + c_n) - 2h_2 \Delta x (c_1 + 2c_2 + \dots + nc_n) \\ &\quad + (\Delta x)^2 (1^2 c_1 + 2^2 c_2 + \dots + n^2 c_n). \end{aligned} \right\} \quad (30)$$

Let  $g_1$  and  $g_2$  be the radii of gyration of the two systems about the places of their first terms, in the same sense as  $r_1$  and  $r_2$  are radii about the centres of parallel forces. Then by the definition,

$$\left. \begin{aligned} S_1 g_1^2 &= (1^2 a_1 + 2^2 a_2 + \dots + m^2 a_m) (\Delta x)^2, \\ S_2 g_2^2 &= (1^2 c_1 + 2^2 c_2 + \dots + n^2 c_n) (\Delta x)^2. \end{aligned} \right\} \quad (31)$$

Substituting in (30), we find with the help of (18) and (19),

$$r_1^2 = g_1^2 - h_1^2, \quad r_2^2 = g_2^2 - h_2^2. \quad (32)$$

Now when the two polynomials (17) are multiplied together, let  $R$  and  $G$  be the radii of gyration of the coefficients in the product, in the same sense as before, about the centre of parallel forces and the first term respectively. As in (32) for the factors, so here for the product, we shall have

$$R^2 = G^2 - H^2,$$

and taking the value of  $H$  from (20),

$$R^2 = G^2 - (h_1 + h_2)^2. \quad (33)$$

The sum of all the coefficients in the product is  $S_1 S_2$ , and by the definition of what we here call the radius of gyration,  $S_1 S_2 G^2$  is equal to the sum of all the products formed by multiplying each coefficient into the square of its distance from the first term. Hence, supposing the first polynomial in (17) to be multiplied successively by the terms of the second one, we get

$$\begin{aligned} S_1 S_2 G^2 = & c_0(1^2 a_1 + 2^2 a_2 + \dots + m^2 a_m)(\Delta x)^2 \\ & + c_1[1^2 a_0 + 2^2 a_1 + \dots + (m+1)^2 a_m](\Delta x)^2 \\ & + c_2[2^2 a_0 + 3^2 a_1 + \dots + (m+2)^2 a_m](\Delta x)^2 \\ & + \dots \dots \dots \\ & + c_n[n^2 a_0 + (n+1)^2 a_1 + \dots + (n+m)^2 a_m](\Delta x)^2. \end{aligned}$$

In the last or general term, the coefficient of  $c_n(\Delta x)^2$ , by expansion of the binomials, and by virtue of (18), (19) and (31), reduces to

$$S_1 \left[ n^2 + 2n \left( \frac{h_1}{\Delta x} \right) + \left( \frac{g_1}{\Delta x} \right)^2 \right].$$

Assigning to  $n$  the values 0, 1, 2, &c., in succession, we get expressions for the coefficients of  $c_0(\Delta x)^2$ ,  $c_1(\Delta x)^2$ , &c., and thus find

$$\begin{aligned} S_1 S_2 G^2 = & c_0 S_1 g_1^2 + c_1 S_1 [1^2 (\Delta x)^2 + 2.1 h_1 \Delta x + g_1^2] \\ & + c_2 S_1 [2^2 (\Delta x)^2 + 2.2 h_1 \Delta x + g_1^2] \\ & + \dots \dots + c_n S_1 [n^2 (\Delta x)^2 + 2 n h_1 \Delta x + g_1^2], \end{aligned}$$

which we put in the form

$$\begin{aligned} S_2 G^2 = & g_1^2 (c_0 + c_1 + \dots + c_n) + 2 h_1 \Delta x (c_1 + 2 c_2 + \dots + n c_n) \\ & + (\Delta x)^2 (1^2 c_1 + 2^2 c_2 + \dots + n^2 c_n). \end{aligned}$$

By means of (18), (19) and (31) this is reduced to

$$G^2 = g_1^2 + g_2^2 + 2 h_1 h_2,$$

which being substituted in (33), gives

$$R^2 = g_1^2 - h_1^2 + g_2^2 - h_2^2,$$

so that by (32) we have finally

$$R^2 = r_1^2 + r_2^2. \quad (34)$$

Thus it is proved that when two entire polynomials are multiplied

together, the square of the radius of gyration in the product is equal to the sum of the squares of the radii in the two factors, the axis of rotation being at the centre of parallel forces in each case. If this product is multiplied by a third factor, the square of the radius for the new product is equal to the sum of the squares of the radii for the three factors, and so on. We thus establish the general theorem, that if any number of polynomial factors having coefficients either positive or negative, are multiplied together, the square of the radius of gyration for the product is equal to the sum of the squares of the radii for all the factors, the centres of parallel forces being the axes of rotation. As a case under this, we have the property demonstrated in my previous article, that the square of the radius of gyration in the  $k$  power of a polynomial is  $k$  times the square of the radius for the polynomial itself. In the notation of that article,  $b'_2(\Delta x)^2$  and  $b^{(k)}_2(\Delta x)^2$  correspond to  $r_1^2$  and  $R^2$  in the notation used here, and since the sum of the coefficients in each polynomial was supposed to be unity in the former demonstration,  $b'_2(\Delta x)^2$  there corresponds also to the product  $S_1 r_1^2$  here. By the mechanical analogy employed, this product answers to the moment of inertia of the system of coefficients whose sum is  $S_1$ .

Particularly interesting results are those already given at pp. 3, 8 and 22 of my former article, where  $p$  and  $q$  are probabilities whose sum is unity, and the binomial  $p+q$  or  $p+qx$  is seen to have its centre of parallel forces at the distance  $h_1 = q\Delta x$  from the place of its first term  $p$ , while the square of its radius of gyration about this centre is  $r_1^2 = pq(\Delta x)^2$ . Then in the expansion of  $(p+q)^m$  our theorems prove that the lever arm or distance of the centre of parallel forces from the first term is  $H = qm\Delta x$ , and the square of the radius of gyration is  $R^2 = pqm(\Delta x)^2$ . Since the coefficients are all positive, they may be regarded as masses, and the centre of parallel forces becomes their centre of gravity. Whenever  $qm$  is a whole number, it is known to be the rank of the greatest term in the expansion, and our proof that the place of this term is then the centre of gravity, shows that in a series of observations whose probabilities or frequencies are represented by the terms of such an expansion, the most probable value of the observed quantity is the arithmetical mean of all the observed values.\* Even if  $qm$  is not a whole number, the place of the arithmetical mean cannot deviate from the place of the greatest term so far as to reach the place of the next term on either side of the greatest. The propriety of adopting the arithmetical mean as the most probable value, is thus shown without

---

\*Quetelet has stated, in effect, that the arithmetical mean divides the length of the series in the ratio of  $p$  to  $q$ . (*Lettres sur la Théorie des Probabilités*, p. 181.) I am not aware that any writer has hitherto given a demonstration of it.

the aid of the assumption, usually made, that positive and negative errors of equal amount are equally probable. Again,  $R^2$  is obtained by taking the sum of the products formed by multiplying each term of the expansion into the square of its distance from the centre of gravity, and dividing this by the sum of the terms, which is unity. Hence if  $qm$  is an integer,  $R$  is simply the quadratic (mean) error, and whether  $qm$  be an integer or not,  $R$  is the quadratic error in the sense that it is the square root of the mean of the squares of the deviations of the typical set of observations from their arithmetical mean. When  $m$  is made an infinity of the second order, and  $\Delta x$  becomes  $dx$ , the difference, if any, between the places of the greatest term and of the arithmetical mean vanishes, and we have

$$R = dx \sqrt{pqm}, \quad (35)$$

a well known expression for the quadratic mean error. (ANALYST, May, 1879, p. 69.)

Polynomials of two variables are found to have radii of gyration with properties analogous to those in the case of one variable. Using the same notation as in (22) and (23), let a horizontal and a vertical straight line be drawn through the centre of parallel forces, in the plane of  $XY$ , and regard these lines as axes of rotation for each of the two polynomials (22). First, let  $r_1$  and  $r_2$  be their radii of gyration about the horizontal axis, in the sense that  $S_1 r_1^2$  is the sum of the products formed by multiplying each coefficient of the first polynomial into the square of its distance from this axis, and  $S_2 r_2^2$  in like manner for the second polynomial. Let  $s'_0, s'_1, \&c.$ , be the sums of the coefficients in the first, second, &c. rows from the bottom, in the first polynomial, and  $s''_0, s''_1, \&c.$ , in like manner for the second polynomial. Then, denoting the lever arms by  $h_1$  and  $h_2$  as before, we have

$$\left. \begin{aligned} S_1 r_1^2 &= s'_0 h_1^2 + s'_1 (h_1 - \Delta y)^2 + s'_2 (h_1 - 2\Delta y)^2 + \dots + s'_n (h_1 - n\Delta y)^2, \\ S_2 r_2^2 &= s''_0 h_2^2 + s''_1 (h_2 - \Delta y)^2 + s''_2 (h_2 - 2\Delta y)^2 + \dots + s''_q (h_2 - q\Delta y)^2. \end{aligned} \right\} \quad (36)$$

Expanding binomials and arranging the terms,

$$\left. \begin{aligned} S_1 r_1^2 &= h_1^2 (s'_0 + s'_1 + \dots + s'_n) - 2h_1 \Delta y (s'_1 + 2s'_2 + \dots + ns'_n) \\ &\quad + (\Delta y)^2 (1^2 s'_1 + 2^2 s'_2 + \dots + n^2 s'_n), \\ S_2 r_2^2 &= h_2^2 (s''_0 + s''_1 + \dots + s''_q) - 2h_2 \Delta y (s''_1 + 2s''_2 + \dots + qs''_q) \\ &\quad + (\Delta y)^2 (1^2 s''_1 + 2^2 s''_2 + \dots + q^2 s''_q). \end{aligned} \right\} \quad (37)$$

The statical moments of the coefficients of the polynomials about their lower rows or axes of  $X$  will be as in (25),

$$\left. \begin{aligned} S_1 h_1 &= (s'_1 + 2s'_2 + \dots + ns'_n) \Delta y, \\ S_2 h_2 &= (s''_1 + 2s''_2 + \dots + qs''_q) \Delta y. \end{aligned} \right\} \quad (38)$$

Let  $g_1$  and  $g_2$  be the radii of gyration of the two systems of coefficients about their lower rows, in the same sense as  $r_1$  and  $r_2$  are radii about the axis of rotation. Then by the definition,



and by (40) we have finally

$$R^2 = r_1^2 + r_2^2. \quad (42)$$

Thus it appears that when any two entire polynomials in  $x$  and  $y$  are multiplied together, and the horizontal line through the centre of parallel forces is taken as an axis, the square of the radius of gyration of the coefficients about this axis in the product is equal to the sum of the squares of the corresponding radii in the two factors. From the similar situation of the polynomials in the horizontal and the vertical directions, it follows also that if the vertical line through the centre of parallel forces is taken as an axis, denoting by  $v_1$  and  $v_2$  the radii of gyration about it for the two factors, and by  $V$  that for their product, we shall have in like manner

$$V^2 = v_1^2 + v_2^2. \quad (43)$$

The sum of the squares of the distances of any coefficient from the horizontal and vertical axes of rotation, is equal to the square of its distance from the centre of parallel forces. Hence, if we suppose the system of coefficients to rotate in the plane of  $XY$ , about the centre of parallel forces, denoting by  $\rho_1$  and  $\rho_2$  the radii of gyration for the two factors, and by  $\rho$  that for their product, we shall have, by the definition,

$$S_1 \rho_1^2 = S_1 r_1^2 + S_1 v_1^2, \quad S_2 \rho_2^2 = S_2 r_2^2 + S_2 v_2^2, \quad S_1 S_2 \rho^2 = S_1 S_2 R^2 + S_1 S_2 V^2$$

$$\rho_1^2 = r_1^2 + v_1^2, \quad \rho_2^2 = r_2^2 + v_2^2, \quad \rho^2 = R^2 + V^2, \quad (44)$$

whence by virtue of (42) and (43),

$$\rho^2 = \rho_1^2 + \rho_2^2. \quad (45)$$

This shows that the square of the radius of gyration for the product, in the plane of  $XY$  and about the centre of parallel forces, is equal to the sum of the squares of the corresponding radii for the two factors.

The theorems here proved for the product of two factors may evidently be extended to that of three, four, or any number of factors, so that always, whether the axis of rotation is horizontal through the centre of parallel forces, or vertical through that centre, or at right angles to the intersection of these two and to the plane of  $XY$ , the square of the radius of gyration for the product will be equal to the sum of the squares of the corresponding radii for all the factors. If the factors are equal, so that the first polynomial in (22) is to be raised to the  $k$  power as in (28), then, denoting the three radii of gyration for the polynomial by  $r_1$ ,  $v_1$  and  $\rho_1$ , and those of its expansion to the  $k$  power by  $R$ ,  $V$  and  $\rho$ , we shall have

$$R^2 = k r_1^2, \quad V^2 = k v_1^2, \quad \rho^2 = k \rho_1^2. \quad (46)$$

As an illustration of the foregoing properties, we will take two factors, the sums of their terms being  $S_1 = 12$  and  $S_2 = 8$ , and whose coefficients, arranged as in (21), are

— 2	2	4	—2	3
1	—3	5	2	—1
2	1	—1	0	1

2	—1	1
0	3	—2
—1	4	2

(47)

The lever arms of the two systems about their lower rows and their left-hand columns are found as in (25), taking for convenience  $\Delta x$  and  $\Delta y$  as the units of  $x$  and  $y$ ;

$$\begin{aligned}
 12h_1 &= 1 \times 4 + 2 \times 5 = 14, & \therefore h_1 &= \frac{7}{6}, \\
 12k_1 &= 1 \times 0 + 2 \times 8 + 3 \times 0 + 4 \times 3 = 28, & \therefore k_1 &= \frac{7}{3}, \\
 8h_2 &= 1 \times 1 + 2 \times 2 = 5, & \therefore h_2 &= \frac{5}{8}, \\
 8k_2 &= 1 \times 6 + 2 \times 1 = 8, & \therefore k_2 &= 1,
 \end{aligned}$$

and these give the positions of the centres of parallel forces. The squares of the radii of gyration about horizontal and vertical axes through the centre of parallel forces are found as in (36).

$$\begin{aligned}
 12r_1^2 &= 3\left(\frac{7}{6}\right)^2 + 4\left(\frac{1}{6}\right)^2 + 5\left(\frac{5}{6}\right)^2, & \therefore r_1^2 &= \frac{23}{6}, \\
 12v_1^2 &= 1\left(\frac{7}{3}\right)^2 + 0\left(\frac{4}{3}\right)^2 + 8\left(\frac{1}{3}\right)^2 + 0\left(\frac{2}{3}\right)^2 + 3\left(\frac{5}{3}\right)^2, & \therefore v_1^2 &= \frac{11}{9}, \\
 8r_2^2 &= 5\left(\frac{5}{8}\right)^2 + 1\left(\frac{3}{8}\right)^2 + 2\left(\frac{1}{8}\right)^2, & \therefore r_2^2 &= \frac{47}{64}, \\
 8v_2^2 &= 1(1)^2 + 6(0)^2 + 1(1)^2, & \therefore v_2^2 &= \frac{1}{4}.
 \end{aligned}$$

Now when the polynomials (47) are multiplied together, the coefficients of their product are as in (48), and the sum of them all is  $S_1 S_2 = 96$ .

— 4	6	4	— 6	12	— 5	3
2	—13	24	4	—13	16	— 7
6	— 7	—12	45	— 6	0	9
— 1	13	—16	7	21	3	— 4
— 2	7	9	— 2	— 3	4	2

(48)

To get the lever arms of this product we proceed as before :

$$96H = 1 \times 23 + 2 \times 35 + 3 \times 13 + 4 \times 10, \quad \therefore H = \frac{43}{24},$$

$$96K = 1 \times 6 + 2 \times 9 + 3 \times 48 + 4 \times 11 + 5 \times 18 + 6 \times 3, \quad \therefore K = \frac{19}{3},$$

and these give the position of the centre of parallel forces.

For the squares of the radii of gyration about horizontal and vertical axes through this centre, we have

$$96R^2 = 15\left(\frac{43}{24}\right)^2 + 23\left(\frac{19}{24}\right)^2 + 35\left(\frac{5}{24}\right)^2 + 13\left(\frac{29}{24}\right)^2 + 10\left(\frac{53}{24}\right)^2, \quad \therefore R^2 = \frac{791}{576},$$

$$96V^2 = 1\left(\frac{19}{3}\right)^2 + 6\left(\frac{7}{3}\right)^2 + 9\left(\frac{4}{3}\right)^2 + 48\left(\frac{1}{3}\right)^2 + 11\left(\frac{2}{3}\right)^2 + 18\left(\frac{5}{3}\right)^2 + 3\left(\frac{8}{3}\right)^2, \quad \therefore V^2 = \frac{19}{36}.$$

Now comparing the results for the two factors with those for their product, we see that

$$\frac{7}{6} + \frac{5}{8} = \frac{43}{24}, \quad \text{and} \quad \frac{7}{3} + 1 = \frac{10}{3},$$

which are in agreement with (26) and (27); also that

$$\frac{23}{36} + \frac{47}{84} = \frac{79}{168}, \quad \text{and} \quad \frac{11}{9} + \frac{1}{4} = \frac{53}{36},$$

which agree with (42) and (43); so that the calculation fully accords with the theory.

Properties analogous to the foregoing can also be shown to belong to polynomials of three variables,  $x$ ,  $y$ , and  $z$ . Suppose two such polynomials, the sums of whose coefficients are  $S_1$  and  $S_2$ , these coefficients being arranged in each case in the form of a rectangular parallelopiped or block, three adjacent sides of which are taken as the coordinate planes of  $ZX$ ,  $ZY$  and  $XY$ . Of course any such polynomial can be put in this form, for the absent terms, if any, can be supposed to enter with coefficients zero.

Let the exponents of the highest powers of  $x$ ,  $y$  and  $z$  be  $m$ ,  $n$  and  $v$  in the first polynomial, and  $p$ ,  $q$  and  $w$  in the second. Then in the first one the number of terms lying along the edges in the three directions of  $X$ ,  $Y$  and  $Z$  are  $m+1$ ,  $n+1$ ,  $v+1$  respectively, and in the second,  $p+1$ ,  $q+1$ ,  $w+1$  respectively. The total numbers of terms in the two are therefore

$$(m+1)(n+1)(v+1), \quad \text{and} \quad (p+1)(q+1)(w+1).$$

When the two are multiplied together, their product is a polynomial which, being similarly arranged, will have  $m+p+1$ ,  $n+q+1$ , and  $v+w+1$  terms along its edges, the total number of terms in it will be

$$(m+p+1)(n+q+1)(v+w+1),$$

and the sum of all its coefficients will be  $S_1 S_2$ . Let each of the two polynomial factors be divided into sections or slices parallel to one of the coordinate planes, for instance that of  $ZX$ , so that the exponent of  $y$  in the terms of the first, second, third, &c., sections shall be 0, 1, 2, &c., respectively. Also let the sums of the coefficients in these sections be  $s'_0, s'_1, s'_2$ , &c., in the first polynomial, and  $s''_0, s''_1, s''_2$ , &c., in the second one. For convenience of expression, let the product of any coefficient into its perpendicular distance from a plane be called its statical moment with respect to that plane, let its product into the square of that distance be called its moment of inertia with respect to the plane, and let the lever arm and the radius of gyration be defined in accordance with this, so that  $h_1, h_2$  and  $H$  are the lever arms of the coefficients in the two polynomial factors and their product, with respect to the plane of  $ZX$ , while  $g_1, g_2$  and  $G$  are their radii of gyration with respect to that plane. Then, denoting by  $\Delta x, \Delta y$  and  $\Delta z$  the intervals between the places of successive coefficients in the three co-ordinate directions, formulas (24) and (25) will hold good for the new meaning of the

same symbols, and when we suppose the first polynomial to be successively multiplied by the first, second, &c., sections of the second one, we shall find by the same reasoning and formulas as before, that (26) is true in its new sense, that is to say, the lever arm of the product, with respect to its side in the plane of  $ZX$ , is equal to the sum of the lever arms of the two factors with respect to their sides lying in the same plane. The same is evidently true with respect to any other corresponding sides of the two given factors and their product. Hence if any number of polynomials in  $x$ ,  $y$  and  $z$  are multiplied together, the lever arm of the coefficients in the final product, with respect to the plane of one of its sides, is equal to the sum of the lever arms for all the factors with respect to their corresponding sides. So too if a single polynomial, whose lever arms with respect to the three coordinate planes are  $h_1$ ,  $k_1$  and  $l_1$ , is to be raised to the  $k$  power, the corresponding lever arms  $H$ ,  $K$  and  $L$  of the expansion will be equal to  $kh_1$ ,  $kk_1$  and  $kl_1$  respectively. If the polynomial is intersected by three planes parallel to the coordinate planes and at intervals from them equal to the corresponding lever arms, their point of common intersection may be called the centre of forces, and if the polynomial is raised to any power or powers, the centre of forces keeps always the same position, relatively to the sides of the block which the expansion occupies. When all the coefficients are positive, they may be regarded as masses, and the centre of forces becomes their centre of gravity.

Passing now to the radius of gyration, the symbols retaining their new significance, we will also denote by  $r_1$ ,  $r_2$  and  $R$  the radii for the two polynomial factors and their product, with respect to a plane parallel to the plane of  $ZX$  and passing through the centre of forces. We shall find that all the formulas from (36) to (42) hold good in their new meaning, so that  $R^2$  is equal to the sum of  $r_1^2$  and  $r_2^2$ . The same would evidently be true if the radii were taken with respect to planes through the centre of forces and parallel to either  $ZY$  or  $XY$ . Hence if any number of polynomials in  $x$ ,  $y$  and  $z$  are multiplied together, the square of the radius of gyration in the product, taken with respect to a plane through the centre of forces and parallel to a side of the block which the product occupies, is equal to the sum of the squares of the corresponding radii for all the factors. If a polynomial with radius  $r_1$  is raised to the  $k$  power, the square of the corresponding radius in the expansion will be  $kr_1^2$ .

Again, let us define the moment of inertia of a coefficient with respect to a point, as the product of that coefficient into the square of its distance from the point, and define the radius of gyration of a system of coefficients with respect to the point, in accordance with this. Let the radii of two poly-

nomial factors in  $x, y, z$ , with respect to the three coordinate planes through the centre of forces in each, be  $r_1, v_1$  and  $u_1$  for the first and  $r_2, v_2$  and  $u_2$  for the second, and let  $R, V$  and  $U$  be the radii for their product. Also let the radii for the two factors and their product, with respect to the centre of forces in each, be  $\rho_1, \rho_2$  and  $\rho$ . The square of the distance of any coefficient from the centre of forces, is equal to the sum of the squares of its distances from three coordinate planes through that centre. We have then, by reasoning similar to that by which (44) was obtained,

$$\rho_1^2 = r_1^2 + v_1^2 + u_1^2, \quad \rho_2^2 = r_2^2 + v_2^2 + u_2^2, \quad \rho^2 = R^2 + V^2 + U^2. \quad (49)$$

But we have already found

$$R^2 = r_1^2 + r_2^2, \quad V^2 = v_1^2 + v_2^2, \quad U^2 = u_1^2 + u_2^2, \quad (50)$$

whence finally,

$$\rho^2 = \rho_1^2 + \rho_2^2. \quad (51)$$

Thus, in the product of two or any number of polynomials in  $x, y$  and  $z$ , the square of what we call the radius of gyration with respect to the centre of forces, is equal to the sum of the squares of the radii for all the factors with respect to their centres of forces.

(To be concluded in No. 4.)

## *PARALLEL CHORDS IN AN ELLIPSE.*

BY PROF. ASAPH HALL.

LET  $\varphi$  be the eccentric angle of a point on the ellipse, or its eccentric anomaly. The rectangular coordinates of this point are

$$x = a \cos \varphi : \quad y = b \sin \varphi.$$

Let the equation of a right line cutting the ellipse be

$$y = a \cdot x + \beta.$$

Denote by  $\varphi_1$  the eccentric angle where the right line cuts the ellipse, and by  $\varphi_{10}$  the similar angle for the opposite point in which this line meets the ellipse. For a second right line we use  $\varphi_2$  and  $\varphi_{20}$ . Hence we have

$$b \cos \frac{1}{2}(\varphi_{10} + \varphi_1) = -a \cdot a \sin \frac{1}{2}(\varphi_{10} + \varphi_1). \quad \text{etc.}$$

If  $A_1$  be the length of the chord from  $\varphi_1$  to  $\varphi_{10}$ ,

$$A_1^2 = (x_{10} - x_1)^2 + (y_{10} - y_1)^2,$$

or

$$A_1^2 = 4a^2 \cdot (1 + a^2) \cdot \sin^2 \frac{1}{2}(\varphi_{10} + \varphi_1) \cdot \sin^2 \frac{1}{2}(\varphi_{10} - \varphi_1).$$

Similarly

$$A_2^2 = 4a^2 \cdot (1 + a^2) \cdot \sin^2 \frac{1}{2}(\varphi_{20} + \varphi_2) \cdot \sin^2 \frac{1}{2}(\varphi_{20} - \varphi_2),$$